Lecture Notes: 4-4 Indeterminate Forms and L'Hospital's Rule (Part 2)

Warm up Problems: Evaluate the limits below. Justify your steps.

a) $\displaystyle\lim_{x\to 0^+} \frac{\arctan 2x}{x}$ ← form $\frac{0}{0}$

$\overset{H}{=} \displaystyle\lim_{x\to 0^+} \left(\frac{\frac{1}{1+4x^2}\cdot 2}{1} \right)$

$= \displaystyle\lim_{x\to 0^+} \frac{2}{1+4x^2}$ algebra

$= 2$ ✓

b) $\displaystyle\lim_{x\to\infty} x\sin(\pi/x)$ form $\infty\cdot 0$

$= \displaystyle\lim_{x\to\infty} \frac{\sin(\pi/x)}{x^{-1}}$ form $\frac{0}{0}$

$\overset{H}{=} \displaystyle\lim_{x\to\infty} \frac{\cos(\pi/x)\cdot(-\pi x^{-2})}{-1\cdot x^{-2}}$

$= \displaystyle\lim_{x\to\infty} \pi\cos(\pi/x) = \pi$

Question 1: What does it mean to a limit is an *indeterminate form*?

It is not possible to just plug in the "a" value to determine the limit. Also, it is possible for different expressions with the same form to have different limits.

Question 2: List several forms that *are* indeterminate and several that *are not* indeterminate.

<u>indeterminate</u>

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$,

1^∞, 0^0, $0\cdot\infty$

<u>NOT indeterminate</u>

$\infty+\infty$

$\infty\cdot\infty$

1^0, 0^∞, $\frac{1}{\infty}$,

$\frac{1}{0}$, ∞^∞

Practice Problems:

1. $\underbrace{\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)}_{\text{form } \infty-\infty} \overset{\text{get a common denominator}}{=} \underbrace{\displaystyle\lim_{x\to 0^+} \frac{\sin x - x}{x\sin x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \underbrace{\displaystyle\lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}}_{\text{form } \frac{0}{0}}$

$\overset{H}{=} \displaystyle\lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$

QUESTION 3: Simplify the expressions below:

(a) If $y = a^b$, then $\ln y =$ $b \ln a$.

(b) If $\lim_{x\to a} \ln[f(x)] = L$, then $\lim_{x\to a} f(x) =$ e^L.

thinking:

$\lim_{x\to a} \ln(f(x)) = \boxed{\ln\left[\lim_{x\to a} f(x)\right] = L}$ is equivalent to $\boxed{e^L = \lim_{x\to a} f(x)}$

HINT: *Transform* the functions below by taking the natural logarithm of the expression in the limit (like part(a) above). Evaluate the limit of this transformed expression. Finally, use the answer of the transformed expression to obtain the limit of the original expression (like part (b) above).

2. $\underbrace{\lim_{x\to\infty} x^{2/x}}_{\text{form } \infty^0} = \boxed{1}$ ← leave space for the answer

transformed problem:

$y = x^{2/x}$

$\ln y = \frac{2}{x} \ln x = \frac{2\ln x}{x}$ ← using (a)

$\lim_{x\to\infty} \frac{2\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{2 \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$

↑ form $\frac{\infty}{\infty}$

Since $\lim_{x\to\infty} \ln(f(x)) = 0$,

$\lim_{x\to\infty} f(x) = e^0 = 1$.

using (b).

3. $\underbrace{\lim_{x\to 0^+} (1 + \sin(2x))^{1/x}}_{\text{form } 1^\infty} = \boxed{e^2}$ ← leave space for my answer.

transform the problem

$y = [1 + \sin(2x)]^{1/x}$

$\ln y = \frac{1}{x} \ln(1 + \sin(2x)) = \frac{\ln(1+\sin(2x))}{x}$

$\underbrace{\lim_{x\to 0^+} \frac{\ln(1+\sin(2x))}{x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0^+} \frac{1}{1+\sin(2x)} \cdot 2\cos(2x) = \lim_{x\to 0^+} \frac{2\cos(2x)}{1+\sin(2x)} = 2$